ON THE STATISTICAL THEORY OF SCALAR SUBSTANCE TRANSPORT IN INHOMOGENEOUS TURBULENCE

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A statistical approach to the problem of scalar substance transport in inhomogeneous turbulence is considered. Differential equations are derived for the "unknown" moments in the equations for one-point correlations. Approximate expressions for nonisotropic two-point correlation tensors are used to close the equations.

Let us examine one of the possible methods of statistically describing the transport of scalar substance (temperature, passive admixture concentration) for inhomogeneous turbulence in an incompressible fluid on the basis of equations for the one- and two-point correlations of the pulsating quantities, as well as the approximate expressions of nonisotropic two-point correlation tensors for closely disposed points. The method proposed for closing the moment equations, including the scalar substance, is analogous to the method used by the authors to analyze momentum transfer in inhomogeneous turbulence [1].

1. The fundamental equations are: the averaged equation of scalar substance transfer in the form of the diffusion equation [2]

$$\frac{\partial \Gamma}{\partial \tau}+U_k\frac{\partial \Gamma}{\partial x_k}+\frac{\partial}{\partial x_k}\overline{u_k\gamma}=\lambda\Delta_x\Gamma; \tag{1}$$

the equation of pulsating fluxes of the scalar substance [3, 4]

$$\frac{\partial}{\partial \tau}\overline{u_i\gamma}+U_k\frac{\partial}{\partial x_k}\overline{u_i\gamma}+\overline{u_iu_k}\frac{\partial \Gamma}{\partial x_k}+\overline{u_k\gamma}\frac{\partial U_i}{\partial x_k}+\frac{\partial}{\partial x_k}\overline{u_iu_k\gamma}+\frac{1}{\rho}\overline{\gamma\frac{\partial p}{\partial x_i}}-\nu\overline{\gamma\frac{\partial^2 u_i}{\partial x_k^2}}-\lambda\overline{u_i\frac{\partial^2\gamma}{\partial x_k^2}}=0; \tag{2}$$

the equation of the triple correlations in (2)

$$\frac{\partial}{\partial \tau}\overline{u_iu_k\gamma}+U_l\frac{\partial}{\partial x_l}\overline{u_iu_k\gamma}+\overline{u_iu_l\gamma}\frac{\partial U_k}{\partial x_l}+\overline{u_iu_ku_l}\frac{\partial \Gamma}{\partial x_l}$$
$$+\overline{u_lu_k\gamma}\frac{\partial U_i}{\partial x_l}+\frac{\partial}{\partial x_l}\overline{u_iu_ku_l\gamma}-\left(\overline{u_i\gamma}\frac{\partial\overline{u_ku_l}}{\partial x_l}+\overline{u_iu_k}\frac{\partial}{\partial x_l}\overline{u_l\gamma}\right.$$
$$\left.+\overline{u_k\gamma}\frac{\partial}{\partial x_l}\overline{u_iu_l}\right)+\frac{1}{\rho}\left(\overline{u_i\gamma\frac{\partial p}{\partial x_k}}+\overline{u_k\gamma\frac{\partial p}{\partial x_i}}\right)-\nu\left(\overline{u_i\gamma\frac{\partial^2u_k}{\partial x_l^2}}+\overline{u_k\gamma\frac{\partial^2u_i}{\partial x_l^2}}\right)-\lambda\overline{u_iu_k\frac{\partial^2\gamma}{\partial x_l^2}}=0. \tag{3}$$

The fourth moments in (3) are expressed in terms of the second of the Millionshchikov hypotheses [5]

$$\overline{u_iu_ku_l\gamma}=\overline{u_iu_k}\,\overline{u_l\gamma}+\overline{u_iu_l}\,\overline{u_k\gamma}+\overline{u_i\gamma}\,\overline{u_ku_l}. \tag{4}$$

Introducing the new coordinate system [6]

$$\xi_k=(x_k)_B-(x_k)_A,\quad (x_k)_{AB}=\frac{1}{2}\left[(x_k)_A+(x_k)_B\right], \tag{5}$$

let us represent the "unknown" correlations which characterize the change in pulsating fluxes of the scalar substance in (2) and (3) because of molecular effects, as [1]:

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$$\lambda\,\overline{u_i \frac{\partial^2\gamma}{\partial x_k^2}} + \nu\,\overline{\gamma \frac{\partial^2 u_i}{\partial x_k^2}} = \frac{1}{4}\,(\lambda+\nu)\,\Delta_x\overline{u_i\gamma} + (\lambda+\nu)\,(\Delta_\xi\,\overline{u_i\gamma'})_0 + (\lambda-\nu)\,\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial \xi_k}\,\overline{u_i\gamma'}\right)_0. \tag{6}$$

$$\lambda\,\overline{u_iu_k \frac{\partial^2\gamma}{\partial x_l^2}} + \nu\left(\overline{u_i\gamma\,\frac{\partial^2 u_k}{\partial x_l^2}} + \overline{u_k\gamma\,\frac{\partial^2 u_i}{\partial x_l^2}}\right) = \frac{1}{4}\,(2\nu+\lambda)\,\Delta_x\,\overline{u_iu_k\gamma}$$
$$+\,\lambda\,(\Delta_\xi\,\overline{u_iu_k\gamma'})_0 + \nu\left[(\Delta_\xi\,\overline{u_iu_k'\gamma'})_0 + (\Delta_\xi\,\overline{u_ku_i'\gamma'})_0\right]. \tag{7}$$

Let us write the correlations containing the pressure pulsations as [1]:

$$\overline{\gamma\,\frac{\partial p}{\partial x_i}} = \frac{1}{2}\,\frac{\partial}{\partial x_i}\,\overline{\gamma p} + \left(\frac{\partial}{\partial \xi_i}\,\overline{\gamma p'}\right)_0, \tag{8}$$

$$\overline{u_i\gamma\,\frac{\partial p}{\partial x_k}} + \overline{u_k\gamma\,\frac{\partial p}{\partial x_i}} = \frac{1}{2}\left(\frac{\partial}{\partial x_k}\,\overline{u_i\gamma p} + \frac{\partial}{\partial x_i}\,\overline{u_k\gamma p}\right) + \left(\frac{\partial}{\partial \xi_k}\,\overline{u_i\gamma p'}\right)_0 + \left(\frac{\partial}{\partial \xi_i}\,\overline{u_k\gamma p'}\right)_0. \tag{9}$$

Let us derive the differential equations for the one-point correlations $\overline{\gamma p}$ and $\overline{u_i\gamma p}$ in (8) and (9). The initial equation is the Poisson equation for the pressure pulsations [2], from which the following equations for the desired correlations are easily obtained:

$$\frac{1}{4\rho}\,\Delta_x\,\overline{p\gamma} + \frac{1}{\rho}\,(\Delta_\xi\,\overline{p\gamma'})_0 + \frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\,\overline{u_n\gamma}$$
$$-\,2\,\frac{\partial U_m}{\partial x_n}\left(\frac{\partial}{\partial \xi_m}\,\overline{u_n\gamma'}\right)_0 + \frac{1}{4}\cdot\frac{\partial^2}{\partial x_m\partial x_n}\,\overline{u_mu_n\gamma} + \left(\frac{\partial^2}{\partial \xi_m\partial \xi_n}\,\overline{u_mu_n\gamma'}\right)_0 = 0. \tag{10}$$

$$\frac{1}{4\rho}\,\Delta_x\,\overline{u_i\gamma p} - \frac{1}{\rho}\cdot\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial \xi_k}\,\overline{u_i\gamma p'}\right)_0 + \frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\,\overline{u_nu_i\gamma}$$
$$+\,\frac{1}{4}\cdot\frac{\partial^2}{\partial x_m\partial x_n}\,\overline{u_mu_nu_i\gamma} - \frac{1}{2}\cdot\frac{\partial}{\partial x_m}\left(\frac{\partial}{\partial \xi_n}\,\overline{u_mu_nu_i'\gamma'}\right)_0$$
$$-\,\frac{1}{2}\cdot\frac{\partial}{\partial x_n}\left(\frac{\partial}{\partial \xi_m}\,\overline{u_mu_nu_i'\gamma'}\right)_0 + \left(\frac{\partial^2}{\partial \xi_m\partial \xi_n}\,\overline{u_mu_nu_i'\gamma'}\right)_0 = 0. \tag{11}$$

The relationships (6)-(9) and Eqs. (10), (11) contain a number of "unknown" terms which are differential operators in ξ of two-point correlations at $\xi = 0$. To determine them it is sufficient to know the correlation tensors for only closely disposed points rather than for the whole ξ-space. Considering the turbulence between two such considered points to be homogeneous, and investigating the expressions for the nonisotropic correlation tensors near the point $\xi = 0$, it can be shown (the authors carried out such an investigation of nonisotropic correlation tensors of even and odd ranks in [1]), that

$$(\Delta_\xi\,\overline{u_i\gamma'})_0 = 0,\quad \left(\frac{\partial}{\partial \xi_m}\,\overline{u_n\gamma'}\right)_0 = 0,\quad \left(\frac{\partial}{\partial \xi_i}\,\overline{\gamma p'}\right)_0 = 0,$$
$$\left(\frac{\partial}{\partial \xi_m}\,\overline{u_n\gamma p'}\right)_0 = 0,\quad \left(\frac{\partial}{\partial \xi_k}\,\overline{u_ku_iu_m'\gamma'}\right)_0 = 0, \tag{12}$$
$$\left(\frac{\partial^2}{\partial \xi_m\partial \xi_n}\,\overline{u_mu_nu_i'\gamma'}\right)_0 = 0,\quad \overline{u_iu_k\gamma} = \overline{u_ku_i'\gamma'}.$$

Moreover, for homogeneous turbulence we have from (10)

$$\frac{1}{\rho}\,(\Delta_\xi\,\overline{p\gamma'})_0 + \left(\frac{\partial^2}{\partial \xi_m\partial \xi_n}\,\overline{u_mu_n\gamma'}\right)_0 = 0. \tag{13}$$

Taking account of (4), (6)-(9), (12), and (13), let us write (2), (3), (10), and (11) as

$$\frac{\partial}{\partial \tau}\,\overline{u_i\gamma} + U_k\,\frac{\partial}{\partial x_k}\,\overline{u_i\gamma} + \overline{u_iu_k}\,\frac{\partial \Gamma}{\partial x_k} + \overline{u_k\gamma}\,\frac{\partial U_i}{\partial x_k}$$
$$+\,\frac{\partial}{\partial x_k}\,\overline{u_iu_k\gamma} + \frac{1}{2\rho}\cdot\frac{\partial}{\partial x_i}\,\overline{\gamma p} - \frac{1}{4}\,(\lambda+\nu)\,\Delta_x\,\overline{u_i\gamma} = 0, \tag{14}$$

$$\frac{\partial}{\partial \tau}\overline{u_iu_k\gamma}+U_l\frac{\partial}{\partial x_l}\overline{u_iu_k\gamma}+\overline{u_iu_l\gamma}\frac{\partial U_k}{\partial x_l}+\overline{u_iu_ku_l}\frac{\partial \Gamma}{\partial x_l}+\overline{u_iu_k\gamma}\frac{\partial U_i}{\partial x_l}+\overline{u_l\gamma}\frac{\partial}{\partial x_l}\overline{u_iu_k}+\overline{u_iu_l}\frac{\partial}{\partial x_l}\overline{u_k\gamma}$$

$$+\overline{u_ku_l}\frac{\partial}{\partial x_l}\overline{u_i\gamma}+\frac{1}{2\rho}\left(\frac{\partial}{\partial x_k}\overline{u_i\gamma p}+\frac{\partial}{\partial x_i}\overline{u_k\gamma p}\right)-\frac{1}{4}(\lambda+2\nu)\Delta_x\overline{u_iu_k\gamma}-\lambda(\Delta_\xi\overline{u_iu_k\gamma'})_0-2\nu(\Delta_\xi\overline{u_iu_k'\gamma'})_0=0, \tag{15}$$

$$\frac{1}{4\rho}\Delta_x\overline{\gamma p}+\frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\overline{u_n\gamma}+\frac{1}{4}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_mu_n\gamma}=0, \tag{16}$$

$$\frac{1}{4\rho}\Delta_x\overline{u_i\gamma p}+\frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\overline{u_nu_i\gamma}+\frac{1}{4}\left(\overline{u_mu_n}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_i\gamma}\right.$$

$$+\overline{u_i\gamma}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_mu_n}+\overline{u_mu_i}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_n\gamma}+\overline{u_n\gamma}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_mu_i}$$

$$+\overline{u_nu_i}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_m\gamma}+\overline{u_m\gamma}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_nu_i}+\frac{\partial\overline{u_mu_n}}{\partial x_m}\cdot\frac{\partial\overline{u_i\gamma}}{\partial x_n}+\frac{\partial\overline{u_i\gamma}}{\partial x_m}\cdot\frac{\partial\overline{u_mu_n}}{\partial x_n}+\frac{\partial\overline{u_mu_i}}{\partial x_m}\cdot\frac{\partial\overline{u_n\gamma}}{\partial x_n}$$

$$\left.+\frac{\partial\overline{u_n\gamma}}{\partial x_m}\cdot\frac{\partial\overline{u_mu_i}}{\partial x_n}+\frac{\partial\overline{u_nu_i}}{\partial x_m}\cdot\frac{\partial\overline{u_m\gamma}}{\partial x_n}+\frac{\partial\overline{u_m\gamma}}{\partial x_m}\cdot\frac{\partial\overline{u_nu_i}}{\partial x_n}\right)=0. \tag{17}$$

Let us examine the "unknown" correlations in (15). It can be shown that the following relationships hold for isotropy

$$(\Delta_\xi Q_{ik,\gamma})_0=\frac{2}{5\sqrt{3}}S_1^*\frac{(-\Delta_\xi Q_{\gamma,\gamma})_0^{1/2}}{\rho_{s,s}^{*1/2}}(\Delta_\xi Q_{i,k})_0,$$

$$(\Delta_\xi Q_{i,k\gamma})_0=\frac{1}{\sqrt{3}}S_2^*\left[\frac{3\overline{\gamma}^2\rho^*_{s,s}-(\Delta_\xi Q_{\gamma,\gamma})_0}{\rho^*_{s,s}}\right]^{1/2}(\Delta_\xi Q_{i,k})_0, \tag{18}$$

where

$$S_1^*=\frac{\overline{\frac{\partial u_r^2}{\partial x_r}\cdot\frac{\partial\gamma}{\partial x_r}}+2\overline{\frac{\partial u_n^2}{\partial x_r}\cdot\frac{\partial\gamma}{\partial x_2}}}{\left[\overline{\left(\frac{\partial u_r^2}{\partial x_r}\right)^2}\right]^{1/2}\left[\overline{\left(\frac{\partial\gamma}{\partial x_r}\right)^2}\right]^{1/2}}=\frac{5\sqrt{3}}{2}\cdot\frac{\rho_{s,s}^{*1/2}}{(-\Delta_\xi Q_{\gamma,\gamma})_0^{1/2}}\cdot\frac{(\Delta_\xi Q_{ss,\gamma})_0}{(\Delta_\xi Q_{s,s})_0},$$

$$S_2^*=\frac{\overline{\frac{\partial u_r}{\partial x_r}\cdot\frac{\partial u_r\gamma}{\partial x_r}}}{\left[\overline{\left(\frac{\partial u_r}{\partial x_r}\right)^2}\right]^{1/2}\left[\overline{\left(\frac{\partial u_r\gamma}{\partial x_r}\right)^2}\right]^{1/2}}=\sqrt{3}\left[\frac{\rho^*_{s,s}}{3\overline{\gamma}^2\rho^*_{s,s}-(\Delta_\xi Q_{\gamma,\gamma})_0}\right]^{1/2}\frac{(\Delta_\xi Q_{s,s\gamma})_0}{(\Delta_\xi Q_{s,s})_0} \tag{19}$$

are dimensionless coefficients which are directly measurable statistical characteristics of the velocity and scalar substance fields.

Taking account of one of the conditions imposed on homogeneous, nonisotropic correlation tensors [1]

$$(L_{np,\ldots,t}\overline{u_iu_j,\ldots,u_m'})_0^*=(L_{np,\ldots,t}Q_{ij,\ldots,m})_0,$$

as well as (18) and (19), we obtain for homogeneous nonisotropic turbulence

$$(\Delta_\xi\overline{u_iu_k\gamma'})_0=\frac{2}{5\sqrt{3}}S_1\frac{(-\Delta_\xi\overline{\gamma\gamma'})_0^{1/2}}{\rho_{s,s}^{1/2}}(\Delta_\xi\overline{u_iu_k'})_0,$$

$$(\Delta_\xi\overline{u_iu_k'\gamma'})_0=\frac{1}{\sqrt{3}}S_2\left[\frac{3\overline{\gamma}^2\rho_{s,s}-(\Delta_\xi\overline{\gamma\gamma'})_0}{\rho_{s,s}}\right]^{1/2}(\Delta_\xi\overline{u_iu_k'})_0, \tag{20}$$

where

$$S_1=\frac{5\sqrt{3}}{2}\cdot\frac{\rho_{s,s}^{1/2}}{(-\Delta_\xi\overline{\gamma\gamma'})_0^{1/2}}\frac{(\Delta_\xi\overline{u_su_s\gamma'})_0}{(\Delta_\xi\overline{u_su_s'})_0},$$

$$S_2=\sqrt{3}\left[\frac{\rho_{s,s}}{3\overline{\gamma}^2\rho_{s,s}-(\Delta_\xi\overline{\gamma\gamma'})_0}\right]^{1/2}\frac{(\Delta_\xi\overline{u_su_s'\gamma'})_0}{(\Delta_\xi\overline{u_su_s'})_0}. \tag{21}$$

It follows from (20) that the system of equations (1), (14)-(17) which describes the transfer of the scalar substance flux in inhomogeneous turbulence is closed to the accuracy of two statistical coefficients S_1 and S_2 (which go over into S_1^* and S_2^* for isotropy), if only the problem of turbulent diffusion of the scalar substance is solved, i.e., the problem of determining the root-mean-square values of the scalar substance $\overline{\gamma^2}$ in a field of inhomogeneous turbulence. Hence, the operator $(-\Delta\overline{\gamma\gamma'})_0$ together with $\overline{\gamma^2}$ defines the microscale of the scalar substance pulsations in homogeneous nonisotropic turbulence

$$\rho_{\gamma,\gamma} = \frac{1}{6\overline{\gamma^2}} (-\Delta_\xi \overline{\gamma\gamma'})_0 , \tag{22}$$

which agrees, for isotropy, with the known microscale of the temperature field [7]

$$\rho^*_{\gamma,\gamma} = \frac{1}{6\overline{\gamma^2}} (-\Delta_r Q_{\gamma,\gamma})_0 = \frac{1}{2}\left(-\frac{\partial^2 R_{\gamma\gamma}}{\partial r^2}\right)_0 = \frac{1}{\lambda_\gamma^2} .$$

2. Let us consider the question of transfer of the root-mean-square scalar substance pulsations in inhomogeneous turbulence. The fundamental equations are: the equation of a double one-point correlation of the scalar substance pulsations [7]

$$\frac{\partial \overline{\gamma^2}}{\partial \tau} + U_k \frac{\partial \overline{\gamma^2}}{\partial x_k} + 2\overline{u_k\gamma} \frac{\partial \Gamma}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_k\gamma^2} - \lambda\Delta_x \overline{\gamma^2} + 2\lambda \overline{\left(\frac{\partial \gamma}{\partial x_k}\right)^2} = 0; \tag{23}$$

the equations of the triple correlations in (23)

$$\frac{\partial}{\partial \tau} \overline{u_k\gamma^2} + U_l \frac{\partial}{\partial x_l} \overline{u_k\gamma^2} + \overline{u_l\gamma^2} \frac{\partial U_k}{\partial x_l} + 2\overline{u_k u_l\gamma} \frac{\partial \Gamma}{\partial x_l}$$

$$+ \frac{\partial}{\partial x_l} \overline{u_k u_l\gamma^2} - \overline{\gamma^2} \frac{\partial \overline{u_k u_l}}{\partial x_l} - 2\overline{u_k\gamma} \frac{\partial \overline{u_l\gamma}}{\partial x_l} + \frac{1}{\rho} \overline{\gamma^2 \frac{\partial p}{\partial x_k}} - \nu \overline{\gamma^2 \frac{\partial^2 u_k}{\partial x_l^2}} - 2\lambda \overline{\gamma u_k \frac{\partial^2 \gamma}{\partial x_l^2}} = 0. \tag{24}$$

Here the correlation $\overline{u_k u_l \gamma}$ is described by (15). The fourth-order moments in (24) are represented in conformity with the Millionshchikov hypothesis, as

$$\overline{u_k u_l \gamma^2} = \overline{u_k u_l}\ \overline{\gamma^2} + 2\,\overline{u_k\gamma}\ \overline{u_l\gamma}. \tag{25}$$

Taking account of the new coordinate system (5), let us write the correlations characterizing the change in $\overline{\gamma^2}$ and $\overline{u_k\gamma^2}$ because of molecular effects in (23) and (24), and the correlation containing the pressure in (24) as

$$2\lambda \overline{\left(\frac{\partial \gamma}{\partial x_k}\right)^2} = \frac{1}{2} \lambda\Delta_x \overline{\gamma^2} - 2\lambda (\Delta_\xi \overline{\gamma\gamma'})_0, \tag{26}$$

$$\nu \overline{\gamma^2 \frac{\partial^2 u_k}{\partial x_l^2}} + 2\lambda \overline{\gamma u_k \frac{\partial^2 \gamma}{\partial x_l^2}} = \frac{1}{4} (2\lambda + \nu)\Delta_x \overline{u_k\gamma^2}$$

$$+ \nu (\Delta_\xi \overline{u_k\gamma'^2})_0 - \nu \frac{\partial}{\partial x_l} \left(\frac{\partial}{\partial \xi_l} \overline{u_k\gamma'^2}\right)_0 + 2\lambda(\Delta_\xi \overline{\gamma\gamma' u_k'})_0 + 2\lambda \frac{\partial}{\partial x_k} \left(\frac{\partial}{\partial \xi_l} \overline{\gamma\gamma' u_k'}\right)_0 , \tag{27}$$

$$\overline{\gamma^2 \frac{\partial p}{\partial x_k}} = \frac{1}{2} \cdot \frac{\partial}{\partial x_k} \overline{\gamma^2 p} + \left(\frac{\partial}{\partial \xi_k} \overline{\gamma^2 p'}\right)_0 . \tag{28}$$

Let us derive the differential equation describing the change in the function $(\Delta_\xi \overline{\gamma\gamma'})_0$ in a field of inhomogeneous turbulence. The starting equation is the dynamic equation of two-point correlation of the scalar substance in inhomogeneous turbulence [8], which is in the coordinates (5)

$$\frac{\partial}{\partial \tau} \overline{\gamma\gamma'} + \overline{u_k\gamma'} \left(\frac{\partial \Gamma}{\partial x_k}\right)_A + \overline{\gamma u_k'} \left(\frac{\partial \Gamma}{\partial x_k}\right)_B$$

$$+ \frac{1}{2} [(U_k)_A + (U_k)_B] \left(\frac{\partial}{\partial x_k}\right)_{AB} \overline{\gamma\gamma'} + [(U_k)_B - (U_k)_A] \frac{\partial}{\partial \xi_k} \overline{\gamma\gamma'}$$

$$+ \frac{1}{2} \left(\frac{\partial}{\partial x_k}\right)_{AB} (\overline{u_k\gamma\gamma'} + \overline{u_k'\gamma'\gamma}) + \frac{\partial}{\partial \xi_k} (\overline{u_k'\gamma'\gamma} - \overline{u_k\gamma\gamma'}) - \frac{1}{2} \lambda (\Delta_x)_{AB} \overline{\gamma\gamma'} - 2\lambda\Delta_\xi \overline{\gamma\gamma'} = 0. \tag{29}$$

Performing the operation $\left[\frac{\partial^2}{\partial\xi_s\partial\xi_p}\ (\ \ )\right]_0$ on (29), we obtain the following equation after simple manipulations associated with introducing the new coordinate system (5):

$$\frac{\partial}{\partial\tau}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0+U_k\frac{\partial}{\partial x_k}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0+\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0\frac{\partial U_k}{\partial x_s}+\left(\frac{\partial^2}{\partial\xi_k\partial\xi_s}\overline{\gamma\gamma'}\right)_0\frac{\partial U_k}{\partial x_p}$$
$$+\frac{1}{4}\frac{\partial^2 U_k}{\partial x_p\partial x_s}\cdot\frac{\partial}{\partial x_k}\overline{\gamma^2}+\frac{1}{2}\overline{u_k\gamma}\frac{\partial^3\Gamma}{\partial x_p\partial x_s\partial x_k}$$
$$-\frac{\partial^2\Gamma}{\partial x_p\partial x_k}\left(\frac{\partial}{\partial\xi_s}\overline{u_k\gamma'}\right)_0-\frac{\partial^2\Gamma}{\partial x_s\partial x_k}\left(\frac{\partial}{\partial\xi_p}\overline{u_k\gamma'}\right)_0+\frac{1}{2}\cdot\frac{\partial}{\partial x_k}\left[\frac{\partial^2}{\partial\xi_s\partial\xi_p}(\overline{u_k\gamma\gamma'}+\overline{u_k'\gamma'\gamma})\right]_0$$
$$+\left[\frac{\partial^3}{\partial\xi_k\partial\xi_s\partial\xi_p}(\overline{u_k'\gamma'\gamma}-\overline{u_k\gamma\gamma'})\right]_0-\frac{1}{2}\lambda\Delta_x\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0-2\lambda\left(\Delta_\xi\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0=0. \quad (30)$$

Furthermore, let us derive the differential equation for the one-point correlation $\overline{\gamma^2 p}$ in (28). This equation is easily obtained from the Poisson equation for the pressure pulsations [1]

$$\frac{1}{4\rho}\Delta_x\overline{p\gamma^2}+\frac{1}{\rho}(\Delta_\xi\overline{p\gamma'\gamma'})_0+\frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\overline{u_n\gamma^2}$$
$$-2\frac{\partial U_m}{\partial x_n}\left(\frac{\partial}{\partial\xi_m}\overline{u_n\gamma'\gamma'}\right)_0-\frac{1}{4}\cdot\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_m u_n\gamma^2}-\left(\frac{\partial^2}{\partial\xi_m\partial\xi_n}\overline{u_m u_n\gamma'\gamma'}\right)_0=0. \quad (31)$$

Using the property of nonisotropic correlation tensors for nearby points [6], we have

$$\left(\frac{\partial}{\partial\xi_m}\overline{u_n\gamma'\gamma'}\right)_0=0,\ \left(\frac{\partial}{\partial\xi_m}\overline{u_n\gamma\gamma'}\right)_0=0,\ \left(\frac{\partial^2}{\partial\xi_m\partial\xi_n}\overline{u_k\gamma'\gamma'}\right)_0=0,$$
$$\left(\frac{\partial^2}{\partial\xi_m\partial\xi_n}\overline{u_k\gamma\gamma'}\right)=0,\ \left(\frac{\partial}{\partial\xi_m}\overline{\gamma^2 p'}\right)_0=0,\ \left(\frac{\partial}{\partial\xi_m}\overline{u_n\gamma'}\right)_0=0, \quad (32)$$
$$\overline{u_m'\gamma'\gamma}=-\overline{u_m\gamma\gamma'}.$$

Moreover, there follows from (31) for homogeneous turbulence

$$\frac{1}{\rho}(\Delta_\xi\overline{p\gamma'\gamma'})_0-\left(\frac{\partial^2}{\partial\xi_m\partial\xi_n}\overline{u_m u_n\gamma'\gamma'}\right)_0=0. \quad (33)$$

Taking account of (25)-(28), (32), and (33), let us write (23), (24), (30), and (31) as

$$\frac{\partial}{\partial\tau}\overline{\gamma^2}+U_k\frac{\partial\overline{\gamma^2}}{\partial x_k}+2\overline{u_k\gamma}\frac{\partial\Gamma}{\partial x_k}+\frac{\partial}{\partial x_k}\overline{u_k\gamma^2}-\frac{1}{2}\lambda\Delta_x\overline{\gamma^2}-2\lambda(\Delta_\xi\overline{\gamma\gamma'})_0=0, \quad (34)$$

$$\frac{\partial}{\partial\tau}\overline{u_k\gamma^2}+U_l\frac{\partial}{\partial x_l}\overline{u_k\gamma^2}+\overline{u_l\gamma^2}\frac{\partial U_k}{\partial x_l}+2\overline{u_k u_l\gamma}\frac{\partial\Gamma}{\partial x_l}$$
$$+\overline{u_k u_l}\frac{\partial\overline{\gamma^2}}{\partial x_l}+2\overline{u_l\gamma}\frac{\partial\overline{u_k\gamma}}{\partial x_l}+\frac{1}{2\rho}\cdot\frac{\partial}{\partial x_k}\overline{\gamma^2 p}-\frac{1}{4}(2\lambda+\nu)\Delta_x\overline{u_k\gamma^2}=0, \quad (35)$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0+U_k\frac{\partial}{\partial x_k}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0+\left(\frac{\partial^2}{\partial\xi_k\partial\xi_p}\overline{\gamma\gamma'}\right)_0\frac{\partial U_k}{\partial x_s}+\left(\frac{\partial^2}{\partial\xi_k\partial\xi_s}\overline{\gamma\gamma'}\right)_0\frac{\partial U_k}{\partial x_p}$$
$$+\frac{1}{4}\cdot\frac{\partial^2 U_k}{\partial x_p\partial x_s}\cdot\frac{\partial}{\partial x_k}\overline{\gamma^2}+\frac{1}{2}\overline{u_k\gamma}\frac{\partial^3\Gamma}{\partial x_s\partial x_p\partial x_k}-$$
$$-2\left(\frac{\partial^3}{\partial\xi_s\partial\xi_p\partial\xi_k}\overline{u_k\gamma\gamma'}\right)_0-\frac{1}{2}\lambda\Delta_x\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0-2\lambda\left(\Delta_\xi\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0=0, \quad (36)$$

$$\frac{1}{4\rho}\Delta_x \overline{p\gamma^2} + \frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\overline{u_n\gamma^2} - \frac{1}{4}\left(\overline{u_m u_n}\frac{\partial^2\overline{\gamma^2}}{\partial x_m \partial x_n} + \frac{\partial\overline{u_m u_n}}{\partial x_n}\cdot\frac{\partial\overline{\gamma^2}}{\partial x_m} + \overline{\gamma^2}\frac{\partial^2\overline{u_m u_n}}{\partial x_m\partial x_n} + \frac{\partial\overline{\gamma^2}}{\partial x_n}\cdot\frac{\partial\overline{u_m u_n}}{\partial x_m}\right)$$

$$-\frac{1}{2}\left(\overline{u_m\gamma}\frac{\partial^2\overline{u_n\gamma}}{\partial x_m\partial x_n} + \frac{\partial\overline{u_m\gamma}}{\partial x_n}\cdot\frac{\partial\overline{u_n\gamma}}{\partial x_m} + \overline{u_n\gamma}\frac{\partial^2\overline{u_m\gamma}}{\partial x_m\partial x_n} + \frac{\partial\overline{u_n\gamma}}{\partial x_n}\cdot\frac{\partial\overline{u_m\gamma}}{\partial x_m}\right) = 0. \tag{37}$$

Let us examine the unknown correlations in (36). It can be shown that the following relationships hold for isotropy:

$$\left(\frac{\partial^3}{\partial\xi_s\partial\xi_p\partial\xi_k}Q_{k\gamma,\gamma}\right)_0 = -\frac{5}{2}S_\gamma^*\sqrt{\overline{u^2}}\rho_{s,s}^{*\,1/2}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}Q_{\gamma,\gamma}\right)_0,$$

$$\left(\Delta_\xi\frac{\partial^2}{\partial\xi_s\partial\xi_p}Q_{\gamma,\gamma}\right)_0 = -\frac{5}{3}\cdot\frac{1}{\lambda}S_\lambda^*\sqrt{\overline{u^2}}\rho_{s,s}^{*\,1/2}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}Q_{\gamma,\gamma}\right)_0, \tag{38}$$

where

$$S_\gamma^* = \frac{\overline{\left(\frac{\partial\gamma}{\partial x_r}\right)^2\frac{\partial u_r}{\partial x_r}}}{\overline{\left(\frac{\partial\gamma}{\partial x_r}\right)^2}\left[\overline{\left(\frac{\partial u_r}{\partial x_r}\right)^2}\right]^{1/2}} = -\frac{2}{5}\cdot\frac{1}{\sqrt{\overline{u^2}}}\cdot\frac{1}{\rho_{s,s}^{*\,1/2}}\frac{\left(\Delta_\xi\frac{\partial}{\partial\xi_k}Q_{k\gamma,\gamma}\right)_0}{(\Delta_\xi Q_{\gamma,\gamma})_0},$$

$$S_\lambda^* = \lambda\frac{\overline{\left(\frac{\partial^2\gamma}{\partial x_r^2}\right)^2}}{\overline{\left(\frac{\partial\gamma}{\partial x_r}\right)^2}\left[\overline{\left(\frac{\partial u_r}{\partial x_r}\right)^2}\right]^{1/2}} = -\frac{3}{5}\cdot\frac{\lambda}{\sqrt{\overline{u^2}}}\cdot\frac{1}{\rho_{s,s}^{*\,1/2}}\cdot\frac{(\Delta_\xi\Delta_\xi Q_{\gamma,\gamma})_0}{(\Delta_\xi Q_{\gamma,\gamma})_0} \tag{39}$$

are dimensionless statistical characteristics of the random velocity and scalar substance fields.

For nonisotropic, homogeneous turbulence we obtain by taking account of (38) and (39):

$$\left(\frac{\partial^3}{\partial\xi_s\partial\xi_p\partial\xi_k}\overline{u_k\gamma\gamma'}\right)_0 = -\frac{5}{2\sqrt{3}}S_\gamma\bar{q}\rho_{s,s}^{1/2}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0,$$

$$\left(\Delta_\xi\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0 = -\frac{5}{3\sqrt{3}}S_\lambda\bar{q}\frac{1}{\lambda}\rho_{s,s}^{1/2}\left(\frac{\partial^2}{\partial\xi_s\partial\xi_p}\overline{\gamma\gamma'}\right)_0, \tag{40}$$

where

$$S_\gamma = -\frac{2\sqrt{3}}{5}\cdot\frac{1}{\bar{q}}\cdot\frac{1}{\rho_{s,s}^{1/2}}\cdot\frac{\left(\Delta_\xi\frac{\partial}{\partial\xi_k}\overline{u_k\gamma\gamma'}\right)_0}{(\Delta_\xi\overline{\gamma\gamma'})_0},$$

$$S_\lambda = -\frac{3\sqrt{3}}{5}\cdot\frac{\lambda}{\bar{q}}\cdot\frac{1}{\rho_{s,s}^{1/2}}\cdot\frac{(\Delta_\xi\Delta_\xi\overline{\gamma\gamma'})_0}{(\Delta_\xi\overline{\gamma\gamma'})_0}. \tag{41}$$

Therefore, the system (34)-(37) which describes the transfer of $\overline{\gamma^2}$ in a field of inhomogeneous turbulence, is closed to the accuracy of the two statistical coefficients S_γ and S_λ which goes over, for isotropy, into the coefficients S_γ^* and S_λ^*.

Let us note that estimated values of the statistical coefficients represented by (19) and (39) can be obtained for isotropic turbulence. Indeed, by using the general inequality which is valid for any random functions f_1 and f_2

$$|\overline{f_1 f_2}| \leqslant \left(\overline{f_1^2}\right)^{1/2}\left(\overline{f_2^2}\right)^{1/2},$$

the coefficients S_1^*, S_2^*, and S_λ^* can be estimated as follows:

$$|S_1^*| \leqslant 1 + 2\sqrt{2},\quad |S_2^*| \leqslant 1,\quad |S_\gamma^*| \leqslant \delta_\gamma^{1/2}, \tag{42}$$

where, as has been shown in [9], the last inequality can be reinforced

$$|S_\gamma^*| \leqslant \frac{2}{3} \delta_\gamma^{1/2} = \frac{2\sqrt{3}}{3}. \quad (43)$$

The second of the coefficients (39) can be estimated by using the first Kolmogorov hypothesis on self-similarity. It hence follows from (36) under the isotropy condition that

$$S_\lambda^* = -\frac{3}{2} S_\gamma^*. \quad (44)$$

The expressions (42), (43), and (44) can also be used as rough estimates of the coefficients S_1, S_2, S_γ, and S_λ of homogeneous nonisotropic turbulence. For a more rigorous determination of the numerical values of these coefficients it is necessary to rely on experimental results on the measurement of the quantities therein.

NOTATION

x_i	are the Cartesian coordinates (i = 1, 2, 3);
τ	is the time;
U_i	is the averaged velocity;
u_i	are the velocity pulsations;
Γ	is the scalar substance (temperature, concentration);
γ	is the scalar substance pulsation;
ρ	is the density;
ν	is the coefficient of kinematic viscosity;
λ	is the coefficient of molecular transfer (temperature conductivity or diffusion);
A, B	are the two-point correlation points;
$(\ )_0$	is the notation of operations at $\xi = 0$;
Δ_x, Δ_ξ	are the Laplace operators in the x_i and ξ_i space, respectively;
$(\ )'$	is the notation of functions at the point B;
$\rho_{i,k}^* = -(1/3)(\partial^2 f/\partial r^2)_0 \delta_{ik}$;	
$f = \overline{u_r u_r'}/\overline{u^2}$	is the longitudinal correlation coefficient;
$g = \overline{u_n u_n'}/\overline{u^2}$	is the transverse correlation coefficient;
u_r, u_n	are the velocity components normal to and along the radius vector between two points;
$Q_{ik,\gamma} = (\overline{u_i u_k \gamma'})^*$;	
f^*	is the notation of functions in isotropy;
$R_{\gamma\gamma} = \overline{\gamma\gamma'}/\overline{\gamma^2}$	is the correlation coefficient of scalar substance pulsations;
$\delta_\gamma = \dfrac{\overline{\left(\frac{\partial\gamma}{\partial x_r}\right)^4}}{\left[\overline{\left(\frac{\partial\gamma}{\partial x_r}\right)^2}\right]^2}$	is the flatness factor of the probability density distribution of the derivatives of the scalar substance pulsations.

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